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# An Extension of AKTH-Theory to Locally Compact Groups (作用素環 の研究とその応用)

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# An extension of AKTH-theory to locally compact groups

by Nobuhiko TATSUUMA

1. Let  $\{\mathcal{A}, G, \alpha\}$  be a  $C^*$ -system. That is,  $\mathcal{A}$  is a  $C^*$ -algebra,  $G$  is a locally compact group, and  $G \ni g \mapsto \alpha_g \in \text{Aut}(\mathcal{A})$  is a continuous homomorphism. Consider an  $\alpha$ -invariant state  $\omega$  on  $\mathcal{A}$ , and the unitary representation  $\{\pi, U_g, \mathcal{H}, \Omega\}$  of  $G$  deduced by GNS-construction.

For any  $A, B \in \mathcal{A}$ , put  $f_{AB}(g) = \omega(B \alpha_g(A)) - \omega(A) \omega(B) = \langle U_g \pi(A) \Omega, \pi(B^*) \Omega \rangle - \langle \pi(A) \Omega, \Omega \rangle \langle \pi(B) \Omega, \Omega \rangle$  and  $g_{AB}(g) = \omega(\alpha_g(A)B) - \omega(A) \omega(B)$ . Then evidently,

$$(1) \quad g_{AB}(g) = f_{A^*B^*}(g) = f_{BA}(g^{-1}).$$

Now we assume the existence of a norm dense  $\alpha$ -invariant  $*$ -subalgebra  $\mathcal{A}_0$  of  $\mathcal{A}$ , for which the followings are valid.

Put

$$\mathcal{F}_0 = (\text{function algebra on } G \text{ algebraically generated by } \{f_{AB}\}_{A, B \in \mathcal{A}_0}),$$

$$\mathcal{F} = (\text{the uniform closure of } \mathcal{F}_0),$$

and construct  $\mathcal{G}_0$  and  $\mathcal{G}$  as same way from  $\{g_{AB}; A, B \in \mathcal{A}_0\}$ .

[Assumption 1]  $\mathcal{F}$  is closed with respect to complex conjugation.

[Assumption 2] For any  $n \geq 1$  and  $A_j, B_j \in \mathcal{A}_0$  ( $j=1, 2, \dots$ ),

$$(2) \quad \int_G \left( \prod_{j=1}^n f_{A_j B_j}(g) - \prod_{j=1}^n g_{A_j B_j}(g) \right) dg = 0.$$

[Assumption 3] There exist  $1 \leq p, q < +\infty$  and a non-zero

element  $f_0 \equiv \sum_k \prod_j f_{A_{j,k}, B_{j,k}} \quad (A_{j,k}, B_{j,k} \in \mathcal{A}_0) \text{ in } \mathcal{F}_0$

such that (i)  $f_0 \in L^p(G)$ ,

(ii)  $g_0 \equiv \sum_k \prod_j g_{A_{j,k}, B_{j,k}} \in L^q(G)$ .

(We use a right Haar measure  $dg$  on  $G$ ).

The purpose of this paper is to show the KMS-property for  $C^*$ -systems which satisfy the above assumptions.

From (1) and [Assumption 1] the following lemma is direct.

Lemma 1  $\mathcal{F} = \mathcal{G}$ , and  $\mathcal{F}$  is closed under the operation

$$f(g) \mapsto f(g^{-1}) .$$

2. We shall give the formulation of our KMS-property on  $C^*$ -systems based on Araki-Kastler-Takesaki-Haag's theory.

When  $G$  is the additive group  $\mathbb{R}$  of real numbers, the ordinary KMS-property is stated as follows.

[KMS] The function  $\psi_{AB}(t) = \omega(B \alpha_t(A))$  can be extended analytically on some strip domain  $\{t; 0 \leq \Im(t) \leq \beta\}$  and

$$\psi_{AB}(t+i\beta) = \omega(\alpha_t(A)B) \text{ for any } t \text{ in } \mathbb{R} \text{ and any } A, B \text{ in } \mathcal{A}.$$

In the other hand for any one-parameter subgroup  $g(t)$  of  $G$ , using the Stone's theorem, we can determine its infinitesimal generator  $iH$ , as  $H$  is a self-adjoint operator on  $\mathcal{H}$  and

$$e^{iHt} = U_{g(t)} .$$

Now in our case, denote by  $K$  the kernel in  $G$  of the homomorphism  $g \rightarrow \alpha_g$ , then our main result is given as follows.

MAIN THEOREM. Under the assumptions 1~3, there exists an one-parameter subgroup  $g(t)$  of  $G/K$ , such that

$$\langle U_g e^{H/2} \pi(A) \Omega, e^{H/2} \pi(B^*) \Omega \rangle = \langle \pi(B) \Omega, U_g \pi(A) \Omega \rangle,$$

for any  $A, B$  in  $\mathcal{A}_0$ .

If the Main Theorem is proved, the function

$$\psi(t) = \omega(B \alpha_{g(t)}(A)) = \langle U_{g(t)} \pi(A) \Omega, \pi(B^*) \Omega \rangle$$

has the analytical extension

$$\psi(t+is) = \langle U_{g(t)} e^{sH/2} \pi(A) \Omega, e^{sH/2} \pi(B^*) \Omega \rangle$$

and  $\psi(t+i) = \langle \pi(B^*) \Omega, U_{g(t)} \pi(A) \Omega \rangle = \omega(\alpha_{g(t)}(A)B)$ .

This shows that the subsystem  $\{\mathcal{A}, R, \alpha_{g(t)}\}$  is just a KMS-C\*-system as originally defined.

3. At first we discuss under slightly more general situation and prove a useful Proposition 1.

Let  $F_0$  be a set of bounded uniformly continuous functions on  $G$ , and  $F$  be the uniform closure of  $F_0$ . For any  $f \in F$ , put  $G_f = \{g \in G; f(gg_1) = f(g_1), \forall g_1 \in G\}$  and  $G_{F_0} = \bigcap_{f \in F_0} G_f$ ,  $G_F = \bigcap_{f \in F} G_f$ .

Lemma 2.  $G_F = G_{F_0}$ , and  $G_F$  is a closed subgroup of  $G$ .

Proof. Because  $f$  is continuous,  $G_f$  is closed. Hence  $G_F, G_{F_0}$  are closed.

For any  $k_1, k_2 \in G_f$ ,  $g \in G$ ,  $f(k_1^{-1}k_2g) = f(k_1(k_1^{-1}k_2g)) = f(k_2g) = f(g)$ . Thus  $k_1^{-1}k_2 \in G_f$ , therefore  $G_f$  and  $G_F, G_{F_0}$  are subgroups.

Obviously  $G_F \subset G_{F_0}$ . If  $G_F \neq G_{F_0}$  there exists  $g_1 \in G_{F_0}$  and  $\notin G_F$ . That is,  $\exists g_2, \exists f \in F$  and  $f(g_1g_2) \neq f(g_2)$  and for  $\forall \varphi \in F_0$ ,  $\varphi(g_1g) =$

$\varphi(g)$ . On the other hand,  $\forall \varepsilon > 0, \exists \varphi_1 \in F_0$  such that  $\|f - \varphi_1\|_\infty < \varepsilon/2$ .  
 Put  $\varepsilon = |f(g_1 g_2) - f(g_2)|$ , then  $|f(g_1 g_2) - f(g_2)| \leq |f(g_1 g_2) - \varphi(g_1 g_2)| + |\varphi(g_1 g_2) - \varphi(g_2)| + |\varphi(g_2) - f(g_2)| < \varepsilon/2 + \varepsilon/2 = \varepsilon = |f(g_1 g_2) - f(g_2)|$ .  
 That is contradiction.

Lemma 3. If there is a non-trivial function  $f_0$  of zero at  $\infty$  in  $F_0$ , then the subgroup  $G_{f_0}$  and  $G_{F_0} = G_F$  are compact.

Proof. If  $G_{f_0}$  is not compact, there exists a sequence  $\{k_j\} \subset G_{f_0}$  such that  $k_j \rightarrow \infty$ . Therefore for some  $g_0 \in G$ , and for all  $j$   $0 \neq f_0(g_0) = f_0(k_j g_0)$ . This contradicts to the assumption for  $f_0$ .

Corollary 1. In such a case,  $L^p(G_F \backslash G)$  is imbedded into  $L^p(G)$  as a space of functions which are constant on  $G_F$ -left cosets.

Hereafter we write  $H = G_F$ .

Lemma 4. If a uniformly continuous function  $f$  on  $G$  belongs to  $L^p(G)$  for some  $p < +\infty$ ,  $f$  is zero at  $\infty$ .

Proof. If  $f$  is not zero at  $\infty$ , there exists a sequence  $\{k_j\} \subset G$  and  $a > 0$  such that  $k_j \rightarrow \infty, |f(k_j)| > a$  for any  $j$ . Uniform continuity of  $f$  asserts the existence of a compact neighborhood  $V$  of  $e$ , such that  $|f(g_1) - f(g_2)| < a/2$  for any  $\forall g_1, g_2$  such that  $g_1 g_2^{-1} \in V$ . Since  $k_j \rightarrow \infty$ , if it is necessary, taking a subsequence, we can assume  $V k_j \cap V k_\ell = \emptyset$  ( $j \neq \ell$ ). Thus,

$$\int_G |f(g)|^p d g \geq \sum_j \int_{V k_j} |f(g)|^p d g \geq \sum_j \int_{V k_j} [|f(k_j)| - (a/2)]^p d g \geq (a/2)^p \sum_j \mu(V) = \infty. \text{ That is contradiction.}$$

Corollary 2. Any  $f \in \mathcal{B}(G) \cap L^p(G)$  ( $p < +\infty$ ) is zero at  $\infty$ .

Here  $\mathcal{B}(G)$  is the ring of functions generated by  $\{ \langle U_g^{\omega} v, u \rangle \}$  ( $\omega$  runs unitary representations of  $G$ , and  $v, u$  run vectors of spaces of representation  $\omega$ ).

Proposition 1. Assume that the above  $F_0$  satisfies the followings.

(i)  $F_0$  is a function algebra, that is, closed under the operations  $+$ ,  $\times$  and scalar multiplication.

(ii)  $F_0$  is invariant under right translations, that is, for any  $f$  in  $F_0$  and any  $g_1$ , the function  $(R_{g_1} f)(g) = f(gg_1)$  of  $g$  is in  $F_0$ .

(iii) The uniform closure  $F$  of  $F_0$  is closed with respect to complex conjugation.

(iv) There exist an  $f_0 (\neq 0)$  in  $F_0$  and  $p < +\infty$ , such that  $f_0 \in L^p(G)$ .

Then there exists a natural number  $n$  and the set

$$F_1 = \left\{ \sum_j^N \varphi_j \cdot R_{g_j}(f_0)^n ; N=1,2,\dots, g_j \in G, \varphi_j \in F_0 \otimes \mathbb{C}1 \right\}$$

in  $F_0 \cap L^1(H \setminus G)$ , is dense in  $L^q(H \setminus G)$  for  $1 \leq q < +\infty$ , and is dense in  $L_c^\infty(H \setminus G) \equiv \{ \text{continuous function of zero at } \infty \text{ on } H \setminus G \}$  with uniform norm.

Proof. If we put  $n = [p] + 1$ ,  $(f_0)^n \in L^1(G) \cap L^\infty(G)$ , therefore  $F_1 \subset F_0 \cap L^1(H \setminus G) \cap L^\infty(H \setminus G)$ . Thus replacing  $f_0^n$  to  $f_0$ , we can consider  $f_0 \in L^1(H \setminus G)$  and  $F_1 \subset L^1(H \setminus G) \cap L^\infty(H \setminus G) \subset L^q(H \setminus G)$  for  $1 \leq q < +\infty$ . And by Lemma 4,  $F_1 \subset L_c^\infty(H \setminus G)$ . Moreover we consider

$$F_2 = \left\{ \sum_j^N \varphi_j \cdot R_{g_j} |f_0|^2 ; N=1,2,\dots, g_j \in G, \varphi_j \in F_0 \otimes \mathbb{C}1 \right\}.$$

In general  $F_2 \not\subset F_0$ , but by the assumption (iii)  $F_2 \subset F$ , since  $\overline{R_{g_j} f_0}$  and therefore  $R_{g_j} |f_0|^2 = (\overline{R_{g_j} f_0})(R_{g_j} f_0)$  are in  $F$ .

Lemma 5. For  $\forall \varphi \in F_2, \forall \varepsilon > 0, 1 \leq p \leq +\infty$ , there exists  $f \in F_1$  such that  $\|\varphi - f\|_p < \varepsilon$ .

Proof. Let  $\varphi = \sum_j^N (\varphi_j \cdot \overline{R_{g_j} f_0}) R_{g_j} f_0 \in F_2$ . Here  $\varphi_j \overline{R_{g_j} f_0} \in F$ , so there exist  $f_j' \in F_0$  such that  $\|\varphi_j \overline{R_{g_j} f_0} - f_j'\|_\infty < (\varepsilon/N \|f_0\|_p)$ . Thus  $\|\varphi - \sum_j f_j' R_{g_j} f_0\|_p < \sum_j \|\varphi_j \overline{R_{g_j} f_0} - f_j'\|_\infty \|R_{g_j} f_0\|_p = \sum_j \|\varphi_j \overline{R_{g_j} f_0} - f_j'\|_\infty \|f_0\|_p < \varepsilon$ .

By the reason of Lemma 5, it is enough to show that  $F_2$  is dense in  $L^q(H \setminus G)$  and  $L_c^\infty(H \setminus G)$ .

Lemma 6.  $F_2$  is (i) a subring of  $F$ , (ii) closed with respect to complex conjugation, (iii) invariant to right translations, (iv)  $F_2 \subset L^1(G) \cap L^\infty(G)$ , so its elements are zero at  $\infty$ , (v) separates any two points  $\tilde{g}_1 \neq \tilde{g}_2$  in  $H \setminus G$ .

Proof.  $F_2$  is the ideal of  $F$  generated by  $\Lambda = \{R_g |f_0|^2; g \in G\}$ , thus (i) is evident. The fact that  $R_g |f_0|^2$  are real-valued, and the assumption (iii) in Proposition 1, give (ii). (iii) is direct result of right invariant properties of  $F_0$ ,  $F$  and  $\Lambda$ .  $R_g |f_0|^2$  are in  $L^1(G)$  and  $F$  is in  $L^\infty(G)$ , hence (iv) is true. At last, if  $f_0(g_1) \neq f_0(g_2)$  then (v) is true for such  $g_1, g_2$ . And if  $0 \neq f_0(g_1 g_0) = f_0(g_2 g_0)$  for some  $g_0$  in  $G$ , by the definition of  $H = G_F$ , there exists a  $\varphi \in F$  such that  $\varphi(g_1) \neq \varphi(g_2)$ , thus  $\varphi \cdot R_{g_0} |f_0|^2$  separates these  $\tilde{g}_1, \tilde{g}_2$ .

Corollary 3. For  $\forall \varphi \in L_c^\infty(H \setminus G), \forall \varepsilon > 0$ , there exists  $f \in F_2$  such that  $\|\varphi - f\|_\infty < \varepsilon$ , that is,  $F_2$  is dense in  $L_c^\infty(H \setminus G)$ .

Proof. Consider the one point compactification space  $X$  of  $H \setminus G$ . We apply the Stone-Weierstrass's theorem to  $F_2 \oplus \mathbb{C}1$  on  $C(X)$ . Thus we get  $f_1 = f + a1 \in F_2 \oplus \mathbb{C}1$  and  $\|\varphi - f_1\|_\infty < \varepsilon/2$ . But  $\varphi$  is zero

at  $\infty$  and  $f \in F_2$  is too. Hence  $|a| < (\varepsilon/2)$ , and  $\|\varphi - f\|_\infty \leq$

$$\|\varphi - f_1\|_\infty + (\varepsilon/2) < \varepsilon.$$

Since  $C_0(H \setminus G) = \{\text{continuous functions on } H \setminus G \text{ with compact supports}\}$  is dense in  $L^p(H \setminus G)$  ( $p < \infty$ ), the following Lemma 7 gives directly a proof of Proposition 1.

Lemma 7. For  $\forall \varphi \in C_0(H \setminus G), \forall \varepsilon > 0, \forall p < +\infty$ , there exists  $f \in F_2$  such that  $\|\varphi - f\|_p < \varepsilon$ .

Proof. Put  $C = [\varphi]$  (support of  $\varphi$ ),  $a = \mu(C)$  (measure of  $C$ ) and  $M = \|\varphi\|_\infty$ . Using the regularity of Haar measure, there exists a relative compact open set  $U$  containing  $C$  such that  $\mu(U) < 2a$ . Moreover we can take a  $\psi \in C_0(H \setminus G)$  such that  $\psi(g) = 1$  for  $g \in C$ , and  $= 0$  for  $g \notin U$ ,  $0 \leq \psi(g) \leq 1 \quad \forall g \in G$ .

By Corollary 3, take  $f_1 \in F_2$  such that

$$\|\varphi - f_1\|_\infty < \rho < \min(1, \varepsilon(2^{p+1}a + 1)^{-1/p}).$$

Put  $m = \int_{G-U} |f_1(g)|^p dg$ , and  $0 < \delta < \min(1, \rho/(M+\rho), \rho m^{-1/p})$ .

Again by Corollary 3, take  $f_2 \in F_2$  such that  $\|\psi - f_2\|_\infty < \delta$  and put  $f = f_1 \cdot f_2$ . Then  $|\varphi(g) - f(g)| = |\varphi(g) - f_1(g)f_2(g)|$  is less than  $|\varphi(g) - f_1(g)| + |1 - f_2(g)| |f_1(g)| < \rho + \delta(M + \rho) < 2\rho$  for  $g \in C$ ,

$$|f_1(g)| |f_2(g)| < \rho(|\psi(g)| + \delta) < \rho(1 + \delta) < 2\rho \quad \text{for } g \in U - C,$$

$$|f_1(g)| |f_2(g)| < |f_1(g)| \delta < \rho m^{-1/p} |f_1(g)| \quad \text{for } g \notin U.$$

Thus  $\|\varphi - f\|_p^p = \int_G |\varphi(g) - f(g)|^p dg = \int_C + \int_{U-C} + \int_{G-U} <$   
 $< 2^p \rho^p \mu(C) + 2^p \rho^p \mu(U-C) + \rho^{p_m-1} \int_{G-U} |f_1(g)|^p dg <$   
 $< (2^{p+1}a + 1) \rho^p \leq \varepsilon^p.$

4. Now we return to our problem concerning to the  $C^*$ -system  $\{\alpha, G, \alpha\}$ . We apply Proposition 1 twice, at first to the case



$F_0 = \mathcal{F}_0$  and second to the case  $F_0 = \mathcal{G}_0$ .

Lemma 8. In both cases,  $G_{F_0}$  ( $= K$ ) are same one and compact normal subgroup of  $G$ .

Proof. If  $F_0 = \mathcal{F}_0$ ,  $K = G_{\mathcal{F}_0} = G_{\mathcal{F}}$ , and if  $F_0 = \mathcal{G}_0$ ,  $K = G_{\mathcal{G}_0} = G_{\mathcal{G}}$ . But by Lemma 1,  $\mathcal{G} = \mathcal{F}$ , thus  $G_{\mathcal{F}} = G_{\mathcal{G}}$ .

For  $\forall k \in K$ ,  $f_{AB}(kg) = f_{AB}(g)$ , that is, for  $\forall g \in G$  and  $\forall A, B \in \mathcal{A}_0$ ,

$$\langle U_k U_g \pi(A) \Omega, \pi(B^*) \Omega \rangle = \langle U_g \pi(A) \Omega, \pi(B^*) \Omega \rangle$$

Thus  $U_k v = v$ , for  $\forall v \in \mathcal{H}$ . This shows  $U_k = I$ , therefore  $K$  is the kernel of this representation, hence normal. [Assumption 3] and Lemma 3, Corollary 2 assure the compactness of  $K$ .

Based on Lemma 8, replacing the factor group  $K \backslash G$  to  $G$ , hereafter we can assume  $K = \{e\}$ . Moreover we take  $p_0 = [\max(p, q)] + 1$  and replace  $f_0^{p_0}, g_0^{p_0}$  to  $f_0, g_0$  in Assumption 3. Thus we can assume that  $f_0, g_0 \in L^1(G) \cap L^\infty(G)$ .

Lemma 9.  $\mathcal{G}_0 = \{ \overline{f_1(g)} ; f_1 \in \mathcal{F}_0 \} = \{ f_1(g^{-1}) ; f_1 \in \mathcal{F}_0 \}$ .

Proof. Since  $\mathcal{A}_0$  is  $*$ -invariant, by (1) we obtain the result.

Proposition 1 leads us to the following lemma.

Lemma 10. The following spaces are dense in  $L^p(G)$  ( $1 \leq p < \infty$ ) and in  $L_c^\infty(G)$ .

$$\begin{aligned} \mathcal{F}_1 &= \left\{ \sum_{j=1}^N f_j(R_{g_j'} f_0) ; N=1,2,\dots, g_j' \in G, f_j \in \mathcal{F}_0 \otimes \mathbb{C}1 \right\}, \\ \mathcal{G}_1 &= \left\{ \sum_{j=1}^N g_j(R_{g_j'} g_0) ; N=1,2,\dots, g_j' \in G, g_j \in \mathcal{G}_0 \otimes \mathbb{C}1 \right\}. \end{aligned}$$

Now define a map  $S$  from  $\mathcal{F}_1$  onto  $\mathcal{G}_1$  by

$$S : \mathcal{F}_1 \ni \sum_k \prod_j f_{A_{j,k} B_{j,k}} \mapsto \sum_k \prod_j g_{A_{j,k} B_{j,k}} \in \mathcal{G}_1.$$

Lemma 11. (i) The map  $S$  is welldefined. That is, for

$\forall f_1 \in \mathcal{F}_1$ ,  $S f_1$  does not depend on the form  $f_1 = \sum \prod f_{A_{j,k} B_{j,k}}$ .

(ii) As a map defined on dense space in  $L^p(G)$  (resp.  $L_c^\infty(G)$ ),  $S$  is closable.

Proof. Summing up the relations (2) in [Assumption 2], we obtain for any  $f_1, f_2$  in  $\mathcal{K}_1$ ,

$$(3) \quad \int_G f_1(g) f_2(g) \, dg = \int_G (Sf_1)(g) (Sf_2)(g) \, dg.$$

If  $f_2$  runs over  $\mathcal{K}_1$ ,  $Sf_2$  runs over  $\mathcal{G}_1$ . Thus if  $f_1 \equiv 0$ ,

$\int_G (Sf_1)(g) k(g) \, dg = 0$  for  $\forall k \in \mathcal{G}_1$ . Because  $\mathcal{G}_1$  is dense in  $L^1(G)$ ,  $Sf_1 \equiv 0$ . This shows,  $S$  is welldefined.

Next if  $f_1 \rightarrow 0$  and  $Sf_1 \rightarrow f_3$  in  $L^p(G)$  (resp.  $L_c^\infty(G)$ ), since  $\mathcal{K}_1 \subset L^q(G)$  ( $(1/p)+(1/q)=1$ ) (resp.  $L^1(G)$ ), the left hand side of (3) tends to zero, and the right hand side tends to

$\int_G f_3(g) Sf_2(g) \, dg$  for any  $f_2$  in  $\mathcal{K}_1$ . Again by the denseness of  $\mathcal{G}_1 = \{Sf_2 ; f_2 \in \mathcal{K}_1\}$  in  $L^q(G)$  (resp. in  $L^1(G)$ ),  $f_3$  must be zero.

Corollary 4. For any  $f_1, f_2 \in \mathcal{K}_1$ ,

$$(4) \quad \langle Sf_1, \overline{Sf_2} \rangle = \langle f_1, \overline{f_2} \rangle.$$

Proof. A direct result of (3).

Let  $T_2$  (resp.  $T_\infty$ ) be the closure of  $S$  as an operator on  $L^2(G)$  (resp.  $L_c^\infty(G)$ ), and  $D_2 \equiv D(T_2)$  (resp.  $D_\infty = D(T_\infty)$ ) be the domains of  $T_2$  (resp.  $T_\infty$ ).

Lemma 12. For  $\forall \varphi \in D_2, \forall \psi \in D_\infty, \psi \cdot \varphi \in D_2$  and

$$T_2(\psi \cdot \varphi) = T_\infty(\psi) \cdot T_2(\varphi).$$

Proof. Let  $\mathcal{K}_1 \ni f_j \rightarrow \varphi, Sf_j \rightarrow T_2(\varphi)$  in  $L^2(G)$ , and

$\mathcal{K}_1 \ni k_j \rightarrow \psi, Sk_j \rightarrow T_\infty(\psi)$  in  $L_c^\infty(G)$ , then  $\mathcal{K}_1 \ni (k_j f_j) \rightarrow \psi \cdot \varphi$ ,  $(Sk_j)(Sf_j) \rightarrow T_\infty(\psi) T_2(\varphi)$  in  $L^2(G)$ . By the definition of  $S$ ,  $(Sk_j)(Sf_j) = S(k_j f_j)$  for  $\forall k_j, f_j \in \mathcal{K}_1$ . Thus we get the result.

Lemma 13.  $S$  commutes with right and left translations  $R_g, L_g$ .

(We use the notations,  $R_g f(g_1) = f(g_1 g)$  and  $L_g f(g_1) = f(g^{-1} g_1)$ .)

Proof. It is enough to show that  $S$  commutes with  $R_g, L_g$  on generators  $\{f_{AB}\}$  of  $\mathcal{F}_1$ . And

$$\begin{aligned} (L_{g_1} R_{g_2} f_{AB})(g) &= \omega(B \alpha_{g_1}^{-1} g g_2(A)) - \omega(A) \omega(B) \\ &= \omega(\alpha_{g_1}(B) \alpha_g(\alpha_{g_2}(A))) - \omega(\alpha_{g_2}(A)) \omega(\alpha_{g_1}(B)) \\ &= f_{\alpha_{g_2}(A), \alpha_{g_1}(B)}(g), \end{aligned}$$

in just same way

$$(L_{g_1} R_{g_2} g_{AB})(g) = g \alpha_{g_2}(A) \alpha_{g_1}(B)(g). \text{ Therefore}$$

$$\begin{aligned} S(L_{g_1} R_{g_2} f_{AB})(g) &= S(f_{\alpha_{g_2}(A) \alpha_{g_1}(B)})(g) = g \alpha_{g_2}(A) \alpha_{g_1}(B)(g) \\ &= (L_{g_1} R_{g_2} g_{AB})(g) = (L_{g_1} R_{g_2} S f_{AB})(g). \end{aligned}$$

Lemma 14. For  $\forall \varphi \in D_2$ ,  $\forall \psi \in L^1(G) \cap L_c^\infty(G)$ , the function

$\langle R_g \varphi, \psi \rangle$  is in  $D_\infty$  and

$$(5) \quad T_\infty(\langle R_g \varphi, \psi \rangle) = \langle R_g T_2 \varphi, \psi \rangle.$$

Proof. For  $\forall f \in \mathcal{F}_1$ ,

$$\begin{aligned} \langle R_g f, \psi \rangle &= \int_G f(g_1 g) \overline{\psi(g_1)} d g_1 = \lim \sum_j^N f(g_j g) \overline{\psi(g_j)} |\Delta_j| \\ &= \lim \sum_j^N (L_{g_j}^{-1} f)(g) \overline{\psi(g_j)} |\Delta_j|. \end{aligned}$$

Because of uniform continuity of  $f, \psi$  and integrability in our case,

this integral converges uniformly in  $g \in G$ . Moreover

$$S(\sum_j (L_{g_j}^{-1} f)(g) \overline{\psi(g_j)} |\Delta_j|) = \sum_j (L_{g_j}^{-1} (Sf))(g) \overline{\psi(g_j)} |\Delta_j|.$$

Thus  $\sum_j f(g_j g) \overline{\psi(g_j)} |\Delta_j|$  and  $S(\sum_j f(g_j g) \overline{\psi(g_j)} |\Delta_j|)$  converge to  $\langle R_g f, \psi \rangle$  and  $\langle R_g Sf, \psi \rangle$  in  $L_c^\infty(G)$  respectively. This shows the results for such a  $f$ .

Next for  $\forall \varphi \in D_2$ , let  $\mathcal{F}_1 \ni f_j \rightarrow \varphi$ ,  $Sf_j \rightarrow T_2 \varphi$  in  $L^2(G)$ ,

then  $\langle R_g f_j, \psi \rangle$  and  $\langle R_g Sf_j, \psi \rangle$  converge to  $\langle R_g \varphi, \psi \rangle$  and

$\langle R_g T_2 \varphi, \psi \rangle$  in  $L_c^\infty(G)$  respectively. That is, the proof is obtained.

Corollary 5. For  $\forall \varphi \in \mathcal{F}_1, \forall \psi \in L^1(G) \cap L_c^\infty(G)$ ,

$$\langle R_g^{-1} \psi, \bar{\varphi} \rangle \in D_\infty, \text{ and } T_\infty(\langle R_g^{-1} \psi, \bar{\varphi} \rangle) = \langle R_g \varphi, \bar{\psi} \rangle$$

(Here  $\bar{\varphi}, \bar{\psi}$  show the complex conjugations of  $\varphi, \psi$  respectively.)

Proof. Indeed,  $\langle R_g^{-1} \psi, \bar{\varphi} \rangle = \overline{\langle R_g \bar{\varphi}, \psi \rangle} = \langle R_g \varphi, \bar{\psi} \rangle$ .

From assumptions,  $\varphi \in \mathcal{F}_1$  and  $\bar{\psi} \in L^1(G) \cap L_c^\infty(G)$ , so lemma 14 leads us to the results.

5. Now we have to discuss the Katz-Takesaki operator on  $G$ , and the relation to the above operator  $T_2$ . We define a unitary operator on  $L^2(G) \otimes L^2(G)$  (called the Katz-Takesaki operator) by

$$(4) \quad W(f_1 \otimes f_2)(g_1, g_2) = f_1(g_1 g_2) f_2(g_2).$$

This operator is closely related with duality theorem as follows.

Proposition 2. The operators  $U \equiv R_g$  of the right regular representation of  $G$ , satisfy

$$(5) \quad W(U \otimes U) = (I \otimes U)W.$$

And conversely, for any non-zero bounded operator  $U$  satisfying (5), there exists unique element  $g$  in  $G$  such that  $R_g = U$ .

For the proof of Proposition 2, we refer [ ].

However for our discussion, we don't need this proposition directly, but the following which is deduced from it.

Proposition 3. For any closed operator  $T$  on  $L^2(G)$  such that

$$(6) \quad W(T \otimes T) = (I \otimes T)W,$$

there exist an element  $g_0$  in  $G$  and an one parameter subgroup  $g(t)$  of  $G$  with infinitesimal generator  $iH$ , such that

$$(7) \quad T = g_0 e^H,$$

(Here we denote the closure of algebraic tensor product of two closed operators  $A$  and  $B$  on  $L^2(G)$  by  $A \otimes B$ .)

Proof. Put  $T^*T = A$ , then  $A$  is a self-adjoint positive definite operator satisfying

$$(8) \quad W(A \otimes A) = (I \otimes A)W.$$

Consider the projection  $P$  onto the space  $\mathcal{H} = (A^{-1}(0))^\perp = \overline{\text{Range}(A)}$ , then by (8)  $P \neq 0$ , and

$$(9) \quad W(P \otimes P) = (I \otimes P)W.$$

Proposition 2 assures that  $P$  is unitary, therefore  $P = I$ . That is  $\mathcal{H} = L^2(G)$ , and we can define the self-adjoint operator  $H = (1/2)\log A$  satisfying

$$(10) \quad W(H \otimes I + I \otimes H) = (I \otimes H)W.$$

Direct calculations show that for  $\forall t \in \mathbb{R}$ ,  $U(t) = e^{iHt}$  is a bounded operator in Proposition 2. Hence we obtain an one-parameter subgroup  $g(t)$  in  $G$  and

$$(11) \quad U(t) = R_{g(t)} \quad \text{for } \forall t \in \mathbb{R}.$$

On the other hand, the bounded operator  $Te^{-H} = U$  satisfies

(5) too. Again Proposition 2 gives an element  $g_0$  in  $G$  such that  $R_{g_0} = U$ . This completes the proof.

We shall call that these operators given in Proposition 3 admissible. In after propositions, we show that the our operator  $T_2$  is admissible.

At first we must remark the following.

Lemma 15. Using any fixed complete orthonormal system  $\{e_\alpha\}$  in  $L^2(G)$ , the Katz-Takesaki operator is expanded as follows.

$$(12) \quad W(f_1 \otimes f_2)(g_1, g_2) = \sum_{\alpha} \varphi_{\alpha}(g_1) \langle R_{g_2} f_1, \varphi_{\alpha} \rangle f_2(g_2).$$

Proof. By only calculation of the expansion.

Lemma 16.  $W(\mathcal{F}_1 \otimes \mathcal{F}_1)$  is in the domain of  $I \otimes T_2$  and

$$(13) \quad (I \otimes T_2)W(f_1 \otimes f_2) = W(Sf_1 \otimes Sf_2) \quad \text{for } \forall f_1, f_2 \in \mathcal{F}_1.$$

Proof. By Schmidt's orthogonalization, we can take all  $\varphi_{\alpha}$ 's in (12) from  $L^1(G) \cap L_c^{\infty}(G)$ . Then by Lemmata 12 and 14, the function  $\varphi_{\alpha}(g_1) \langle R_{g_2} f_1, \varphi_{\alpha} \rangle f_2(g_2)$  are in  $D(I) \otimes D(T_2) \subset D(I \otimes T_2)$  (The domain of  $I \otimes T_2$ ), and

$$\begin{aligned} (14) \quad (I \otimes T_2)(\varphi_{\alpha}(g_1) \langle R_{g_2} f_1, \varphi_{\alpha} \rangle f_2(g_2)) &= \\ &= \varphi_{\alpha}(g_1) T_{\infty}(\langle R_{g_2} f_1, \varphi_{\alpha} \rangle)(T_2 f_2)(g_2) \\ &= \varphi_{\alpha}(g_1) \langle R_{g_2} T_2 f_1, \varphi_{\alpha} \rangle (T_2 f_2)(g_2). \end{aligned}$$

Moreover,  $\sum^N \varphi_{\alpha}(g_1) \langle R_{g_2} f_1, \varphi_{\alpha} \rangle f_2(g_2)$  and  $(I \otimes T_2) \sum^N (\varphi_{\alpha}(g_1) \langle R_{g_2} f_1, \varphi_{\alpha} \rangle f_2(g_2)) = \sum^N \varphi_{\alpha}(g_1) \langle R_{g_2} T_2 f_1, \varphi_{\alpha} \rangle (T_2 f_2)(g_2)$  converge to  $W(f_1 \otimes f_2)(g_1, g_2)$  and  $W(Sf_1 \otimes Sf_2)$  in  $L^2(G) \otimes L^2(G)$  respectively. This gives the results.

Lemma 17.  $W^{-1}(\mathcal{F}_1 \otimes \mathcal{F}_1)$  is in the domain of  $T_2 \otimes T_2$  and

$$(15) \quad (T_2 \otimes T_2)W^{-1}(f_1 \otimes f_2) = W^{-1}(I \otimes S)(f_1 \otimes f_2) \quad \text{for } \forall f_1, f_2 \in \mathcal{F}_1.$$

Proof.  $W^{-1}$  is given by  $W^{-1}(f_1 \otimes f_2)(g_1, g_2) = f_1(g_1 g_2^{-1}) f_2(g_2)$ .

Thus for  $\forall f_1, f_2 \in \mathcal{F}_1$ ,  $(S \otimes I)W^{-1}(f_1 \otimes f_2)(g_1, g_2) = (S \otimes I)(R_{g_2}^{-1} f_1(g_1) f_2(g_2)) = (S R_{g_2}^{-1} f_1)(g_1) f_2(g_2) = (R_{g_2}^{-1} S f_1)(g_1) f_2(g_2) =$   
 $= (S f_1)(g_1 g_2^{-1}) f_2(g_2) = \sum \varphi_{\alpha}(g_1) \langle R_{g_2}^{-1}(S f_1), \varphi_{\alpha} \rangle f_2(g_2).$

This shows that  $W^{-1}(\mathcal{F}_1 \otimes \mathcal{F}_1)$  is in  $D(\overline{S \otimes I}) = D(T_2 \otimes I)$ .

Next we shall show that  $(S \otimes I)W^{-1}(\mathcal{F}_1 \otimes \mathcal{F}_1)$  is in  $D(I \otimes T_2)$ .

Indeed by Corollary 4 and the fact  $Sf_1 \in \mathcal{G}_1 \subset L^1(G) \cap L_c^{\infty}(G)$ , if we select the C.O.N.S  $\{\varphi_{\alpha}\}$  as  $\overline{\varphi_{\alpha}} \in \mathcal{F}_1$ ,  $\langle R_{g_2}^{-1}(Sf_1), \varphi_{\alpha} \rangle \in D_{\infty}$  and hence  $\varphi_{\alpha}(g_1) \langle R_{g_2}^{-1}(Sf_1), \varphi_{\alpha} \rangle f_2(g_2) \in D(I \otimes T_2)$ .

Using (4),  $(I \otimes T_2)(\varphi_\alpha(g_1) \langle R_{g_2}^{-1}(Sf_1), \varphi_\alpha \rangle f_2(g_2)) =$

$$\begin{aligned} & \varphi_\alpha(g_1) \langle R_{g_2} S \overline{\varphi_\alpha}, \overline{Sf_1} \rangle (Sf_2)(g_2) = \varphi_\alpha(g_1) \langle S(R_{g_2} \overline{\varphi_\alpha}), \overline{Sf_1} \rangle (Sf_2)(g_2) = \\ & = \varphi_\alpha(g_1) \langle R_{g_2} \overline{\varphi_\alpha}, \overline{f_1} \rangle (Sf_2)(g_2) = \varphi_\alpha(g_1) \langle R_{g_2}^{-1} f_1, \varphi_\alpha \rangle (Sf_2)(g_2). \end{aligned}$$

Obviously  $\sum_{\alpha}^N \varphi_\alpha(g_1) \langle R_{g_2}^{-1}(Sf_1), \varphi_\alpha \rangle f_2(g_2)$  and

$\sum_{\alpha}^N \varphi_\alpha(g_1) \langle R_{g_2}^{-1} f_1, \varphi_\alpha \rangle (Sf_2)(g_2) (= (I \otimes T_2)(\sum_{\alpha}^N \varphi_\alpha(g_1) \langle R_{g_2}^{-1}(Sf_1), \varphi_\alpha \rangle \times f_2(g_2)))$  converge to  $(S \otimes I)W^{-1}(f_1 \otimes f_2)$  and  $W^{-1}(f_1 \otimes Sf_2)$  in  $L^2(G) \otimes L^2(G)$  respectively. That is,  $(S \otimes I)W^{-1}(f_1 \otimes f_2) \in D(I \otimes T_2)$  and  $(I \otimes T_2)(S \otimes I)W^{-1}(f_1 \otimes f_2) = W^{-1}(f_1 \otimes Sf_2)$ . Combining these, we get the wanted results.

Summalizing Lemmata 16 and 17, we conclude,

Proposition 4. The closed operator  $T_2$  is admissible.

Now we are in the step to apply Proposition 3 together with Lemma 13 to our operator  $T_2$ , and get,

Lemma 18. There exist an element  $g_0$  with order 1 or 2 and an one-parameter subgroup  $g(t)$  in the centre  $Z(G)$  of  $G$  such that  $T_2 = R_{g_0} e^H$ , Here  $iH$  is the infinitesimal generator of  $R_{g(t)}$ .

Proof. The existence of  $g_0$  and  $g(t)$  are an direct results of the above arguments, so we must show that  $g_0$  and  $g(t)$  are in  $Z(G)$  and the order of  $g_0$  is atmost two.

But because (7) gives the polar decomposition of  $T_2$ , and by Lemma 13,  $T_2$  hence  $R_{g_0}$  and  $e^H$  must commute with  $R_g$  ( $\forall g \in G$ ).

The relation (3) and the definitions of  $\mathcal{F}_1, \mathcal{G}_1$  and  $S$  give

$\langle f_1, Sf_2 \rangle = \langle Sf_1, f_2 \rangle$  for  $f_1, f_2 \in \mathcal{F}_1$ . This concludes  $T_2$  is symmetric. But since  $R_{g_0}$  is unitary and  $e^H$  is positive definite without kernel,  $R_{g_0}$  must be the form  $P-(I-P)$  for some projection  $P$ .

Hence  $(R_{g_0})^2 = I$ , and  $g_0^2 = e$ .

The assertion of Lemma 18 talks about only operators on  $L^2(G)$ . However using [Assumption 2], we can extend this to the whole space as follows. That is, consider the operators on  $\mathcal{H}$ ,

$H_0 \equiv (1/i)(d/dt)U_{g(t)} \big|_{t=0}$ ,  $V \equiv e^{H_0}$  and  $T^1 \equiv U_{g_0} V$  in which  $g_0, g(t)$  are elements of  $G$  given in Lemma 18.

Lemma 19.  $\langle \pi(B)\Omega, U_g \pi(A^*)\Omega \rangle = \langle U_{g_0} V^{1/2} \pi(A)\Omega, V^{1/2} \pi(B^*)\Omega \rangle$   
for  $\forall A, B \in \mathcal{A}_0$ .

Proof. Let  $\varphi(t) \equiv e^{-t^2}$  and for  $c \in (0, \infty)$  and  $A \in \mathcal{A}$ ,

$$A_{\varphi, c} \equiv (2c/\sqrt{\pi}) \int_{-\infty}^{\infty} \alpha_{g(t)}(A) \varphi(ct) dt.$$

Then it is easy to see  $\pi(A_{\varphi, c})\Omega \in D(T^1)$  and  $A_{\varphi, c} \xrightarrow{c \rightarrow \infty} A$  in  $\mathcal{A}$ , hence  $\pi(A_{\varphi, c})\Omega \rightarrow \pi(A)\Omega$  and  $\pi(A_{\varphi, c}^*)\Omega \rightarrow \pi(A^*)\Omega$  in  $\mathcal{H}$ . Denote  $\mathcal{A}_1 \equiv \{A_{\varphi, c}; c \in (0, \infty), A \in \mathcal{A}_0\}$ . Then direct calculations lead us to

$$R_{g_0} \left( \sum_n \frac{1}{n!} \left( \frac{1}{i} \frac{d}{dt} \right)^n (R_{g(t)} f) \bigg|_{t=0} \right)(g) = \langle U_g T^1 \pi(A)\Omega, \pi(B^*)\Omega \rangle - \langle \pi(A)\Omega, \Omega \rangle \langle \pi(B)\Omega, \Omega \rangle.$$

Since the convergences of  $H_0 \pi(A)\Omega = (1/i) \lim_{t \rightarrow 0} t^{-1} (U_{g(t)} \pi(A)\Omega - \pi(A)\Omega)$  and  $V \pi(A)\Omega = \sum_n (1/n!) H_0^n \pi(A)\Omega$  are in norm sense, the convergence of the left hand side is uniform in  $g$ . Generally  $\mathcal{A}_1$  is not contained in  $\mathcal{A}_0$ , but all elements of  $\mathcal{A}_1$  are norm limits of elements of

$\mathcal{A}_0$  and vice versa. Hence  $f_{A,B}, g_{A,B}$  ( $A, B \in \mathcal{A}_0 \cup \mathcal{A}_1$ ) are uniform limits of  $f_{A_j, B_j}, g_{A_j, B_j}$  ( $A_j, B_j \in \mathcal{A}_0$ ). And by (2)

$$(16) \quad \int (f_{AB}(g) f_1(g)) f_2(g) dg = \int (g_{AB}(g) S f_1(g)) S f_2(g) dg$$

for  $\forall A, B \in \mathcal{A}_1 \cup \mathcal{A}_0, \forall f_1, f_2 \in \mathcal{F}_1$ .



Now  $f_1 \ni f_{A_j B_j} f_1 \rightarrow f_{AB} f_1$ ,  $T_2(f_{A_j B_j} f_1) = (Sf_{A_j B_j})(Sf_1) = g_{A_j B_j}(Sf_1) \rightarrow g_{AB}(Sf_1)$  in  $L^2(G)$ , therefore  $f_{AB} f_1 \in D_2$  and  $T_2(f_{AB} f_1) = g_{AB} Sf_1$  for  $\forall A, B \in \mathcal{A}_1 \cup \mathcal{A}_0$ . And Lemma 18 assures  $T_2(f_{AB} f_1) = R_{g_0}(\sum_n (1/n!)) (-i)^n (d/dt)^n (R_{g(t)} f_{AB}) (Sf_1)$  for  $\forall A \in \mathcal{A}_1$ , thus  $g_{AB} = R_{g_0}(\sum_n (1/n!)) (-i)^n (d/dt)^n (R_{g(t)} f_{AB})$  (converges in  $L^\infty(G)$ ). Therefore

$$\langle U_g T^1 \pi(A) \Omega, \pi(B^*) \Omega \rangle = \langle \pi(B) \Omega, U_g \pi(A^*) \Omega \rangle \text{ for } \forall A \in \mathcal{A}_1,$$

$\forall B \in \mathcal{A}$ . Especially for  $A, B \in \mathcal{A}_0$ ,  $A_{\varphi, c} B_{\varphi, c} \in \mathcal{A}_1$ , hence

$$\langle U_{g_0} V^{1/2} \pi(A_{\varphi, c_1}) \Omega, V^{1/2} \pi(B_{\varphi, c_2}^*) \Omega \rangle = \langle \pi(B_{\varphi, c_2}) \Omega, U_g \pi(A_{\varphi, c_1}^*) \Omega \rangle.$$

Put  $g = g_0^{-1}$ . When  $c$  tends to  $\infty$ ,  $A_{\varphi, c} \rightarrow A$  in  $\mathcal{A}$ ,  $A_{\varphi, c}^* \rightarrow A^*$ ,

$\pi(A_{\varphi, c}) \Omega \rightarrow \pi(A) \Omega$ , and  $\pi(A_{\varphi, c}^*) \Omega \rightarrow \pi(A^*) \Omega$ . Taking the limit,

$$\begin{aligned} \text{we obtain } \lim_{\substack{c_1 \rightarrow \infty \\ c_2 \rightarrow \infty}} \langle V^{1/2} \pi(A_{\varphi, c_1}) \Omega, V^{1/2} \pi(B_{\varphi, c_2}^*) \Omega \rangle &= \\ &= \langle \pi(B) \Omega, U_{g_0}^{-1} \pi(A^*) \Omega \rangle. \text{ Hence} \end{aligned}$$

$$\begin{aligned} \lim_{\substack{c_1 \rightarrow \infty \\ c_2 \rightarrow \infty}} \|V^{1/2} \pi(A_{\varphi, c_1}) \Omega - V^{1/2} \pi(A_{\varphi, c_2}) \Omega\|^2 &= \lim_{\substack{c_1 \rightarrow \infty \\ c_2 \rightarrow \infty}} \sum_{i, j=1, 2} (-1)^{i+j} \times \\ &\times \langle V^{1/2} \pi(A_{\varphi, c_1}) \Omega, V^{1/2} \pi(A_{\varphi, c_2}) \Omega \rangle = \\ &= \sum_{i, j=1, 2} (-1)^{i+j} \langle \pi(A) \Omega, U_{g_0}^{-1} \pi(A) \Omega \rangle = 0. \end{aligned}$$

That is,  $\{V^{1/2} \pi(A_{\varphi, c}) \Omega\}_{c \rightarrow \infty}$  is a Cauchy sequence in  $\mathcal{H}$ , so

$\pi(A) \Omega \in D(V^{1/2})$ ,  $V^{1/2} \pi(A) \Omega = \lim_{c \rightarrow \infty} V^{1/2} \pi(A_{\varphi, c}) \Omega$ , and

$$\begin{aligned} \langle U_{g_0} V^{1/2} \pi(A) \Omega, V^{1/2} \pi(B^*) \Omega \rangle &= \\ &= \lim_{c \rightarrow \infty} \langle U_{g_0} V^{1/2} \pi(A_{\varphi, c}) \Omega, V^{1/2} \pi(B_{\varphi, c}^*) \Omega \rangle \\ &= \langle \pi(B) \Omega, U_g \pi(A^*) \Omega \rangle = g_{AB}(g). \end{aligned}$$

$\mathcal{A}_0$  is norm dense in  $\mathcal{A}$ , therefore  $\mathcal{A}_1$  is norm dense in  $\mathcal{A}$ ,

and  $\{\pi(A) \Omega; A \in \mathcal{A}_1\}$  is dense in  $\mathcal{H}$  too. Thus,

Corollary 6. There exists a norm dense subalgebra  $\mathcal{A}_1$  in

$\mathcal{A}$ , and a closed operator  $T^1$  on  $\mathcal{H}$  such that  $\{\pi(A) \Omega; A \in \mathcal{A}_1\} \subset D(T^1)$ ,

and  $\langle U_g T^1 \pi(A) \Omega, \pi(B^*) \Omega \rangle = \langle \pi(B) \Omega, U_g \pi(A^*) \Omega \rangle$   
for  $\forall g \in G, \forall A \in \mathcal{A}_1, \forall B \in \mathcal{A}$ .

Lemma 20. In Lemma 18, the element  $g_0$  is equal to  $e$ .

Proof. Consider two positive definite functions

$$\begin{aligned}\psi_1(g) &= \langle U_g \pi(B) \Omega, \pi(B) \Omega \rangle, \\ \psi_2(g) &= \langle U_g v^{1/2} \pi(B^*) \Omega, v^{1/2} \pi(B^*) \Omega \rangle.\end{aligned}$$

In Lemma 19, putting  $A = B^*$ , we obtain

$$(17) \quad \psi_1(g) = \psi_2(gg_0) \quad \text{for } \forall g \in G.$$

But,  $\psi_1(g_0^{-1}) = \psi_2(e) = \|\psi_2\|_\infty = \|\psi_1\|_\infty = \psi_1(e)$ , therefore  
 $\langle U_{g_0} \pi(B) \Omega, \pi(B) \Omega \rangle = \|\pi(B) \Omega\|^2$  and  $U_{g_0} \pi(B) \Omega = \pi(B) \Omega$   
for  $\forall B \in \mathcal{A}_0$ . That is  $U_{g_0} = I$ , hence  $g_0$  is in  $K = \{e\}$ .

Thus the results of Lemmata 18 - 20 give a proof of our Main theorem.

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